24^{th} BALKAN MATHEMATICAL OLYMPIAD
Rhodes, Hellas (April 28, 2007)

Problem 2.

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)+y) = f(f(x)-y) + 4f(x)y \quad \text{for any } x, y \in \mathbb{R}.$$

Solution. It is clear that the function $f \equiv 0$ satisfies the given condition.

Assume that $f \not\equiv 0$. Choose x_0 such that $f(x_0) \neq 0$ and set $\widetilde{y} = \dfrac{y}{4f(x_0)}$ for any y. Then plugging in to the given relation x_0 for x and $\widetilde{y}$ for y we get

$$y = f(f(x_0)+\widetilde{y}) - f(f(x_0)-\widetilde{y}). \tag{1}$$

For any y_1, y_2, plugging in to (1) $\dfrac{y_1-y_2}{2}$ for y, we get

$$\frac{y_1-y_2}{2} = f(f(x_0) + \widetilde{\frac{y_1-y_2}{2}}) - f(f(x_0) - \widetilde{\frac{y_1-y_2}{2}}).$$

In other words, for any y_1, y_2 there exist $x_1(y_1,y_2) = f(x_0) + \widetilde{\frac{y_1-y_2}{2}}$, $x_2(y_1,y_2) = f(x_0) + \widetilde{\frac{y_1-y_2}{2}}$ such that $\dfrac{y_1-y_2}{2} = f(x_1) - f(x_2)$, i.e. such that

$$2f(x_1) - y_1 = 2f(x_2) - y_2. \tag{2}$$

On the other hand, replacing y by $f(x)-y$ in the given condition gives

$$f(2f(x)-y) = f(y) + 4f(x)(f(x)-y), \text{ i.e.}$$

$$f(y) - y^2 = f(2f(x)-y) - (2(f(x)-y)^2. \tag{3}$$

Now if for any two y_1, y_2, we plug in to (3), $x_1(y_1,y_2)$ and $x_2(y_1,y_2)$ respectively, we get

$$f(y_1) - y_1^2 = f(2f(x_1)-y_1) - (2(f(x_1)-y_1)^2$$

and

$$f(y_2) - y_2^2 = f(2f(x_2)-y_2) - (2(f(x_2)-y_2)^2$$

Then by (2) we get $f(y_1) - y_1^2 = f(y_2) - y_2^2$. Since this happens for any two y_1, y_2, we conclude that $f(x) - x^2 =$ constant for all x, thus $f(x) = x^2 + c, c \in \mathbb{R}$. It is easy to check that such a function satisfies the condition of the problem. ■