24th BALKAN MATHEMATICAL OLYMPIAD
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Problem 4. For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex n-gons in the plane such that the sets $C_1 \cap C_2$, $C_2 \cap C_3$, $C_3 \cap C_1$ are finite. Find the maximum number of points of the set $C_1 \cap C_2 \cap C_3$.

Solution 1: Let us first observe that, if a line intersects a convex $n-$gon at finitely many points, then the number of such points is at most 2. Therefore any two of the $n-$gons may intersect in at most $2n$ points. Choose two of the $n-$gons, C_1, C_2, and say that their intersection points are $p_1, p_2, \ldots, p_k$. Thus $k \leq 2n$. Say that the union of the set of vertices of C_1 and C_2 is $\{q_1, q_2, \ldots, q_{2n}\}$. We note that it is possible to have $q_i = q_j$ for some $i \neq j$.

We will define a one-to-one function f from $\{p_1, p_2, \ldots, p_k\}$ to $\{q_1, q_2, \ldots, q_{2n}\}$ as follows. First of all, orient all $n-$gons in the clockwise direction. Thus, if one traverses an $n-$gon according to this orientation, the interior is on the right and the exterior is on the left. For every p_i, there exist precisely two line segments (of non-zero length) which are subsets of C_1 or C_2, say $[q_j, p_i]$ on C_1 and $[q_k, p_i]$ on C_2, such that one can approach to p_i via these line segments in the clockwise direction. Suppose, that the two vectors $p_i - q_j$ and $p_i - q_k$, in this order, form a right handed coordinate system. Then none of the points on $[q_j, p_i]$ can be on or in the interior of C_2, since for any point q on or in the interior of C_2, the vectors $p_i - q$ and $p_i - q_k$ are either positive multiples of each other, or form a left handed coordinate system. In this case we set $f(p_i) = q_j$. Otherwise we set $f(p_i) = q_k$. In both cases, the argument above shows that there are no other intersection points between $f(p_i)$ and p_i, in the clockwise direction. Let us now show that f is $1-1$. If $f(p_i) = f(p_l) = q$ and q say (without loss of generality) belong to C_1, then the first intersection point encountered when one starts from q and traverses C_1 in the clockwise direction has to be both p_i and p_l, hence $p_i = p_l$.

Now let us estimate the number of p_i's that can be contained by the third polygon C_3. Each edge of C_3 contains exactly $0, 1$ or 2 of the p_i's. Suppose that a given edge of C_3 contains 2 of the p_i's, say p_1 and p_2. Since C_1 and C_2 are convex and their intersection with C_3 is generic, they should have vertices between (in the clockwise sense) p_1 and p_2 (with this order), and outside C_3. We claim that at least one of these vertices is not in the set $f(C_1 \cap C_2 \cap C_3)$. Let $q_1 \in C_1$ and $q_2 \in C_2$ respectively be between (in the clockwise sense) p_1 and p_2 (with this order). If q_1 is on or in the interior of C_2 (or q_2 is on or in the interior of C_1), then q_1 (or q_2) is not in the image of f, since recall that $f(p)$ for any $p \in C_1 \cap C_2$ (thus for any $p \in C_1 \cap C_2 \cap C_3$) is a point of one of C_1, C_2 not on or in the interior of the other. So the claim is established in this case. The remaining case is to assume that none of the vertices of any of C_1, C_2 that lie outside C_3 (and between (in the clockwise sense) p_1 and p_2

(with this order)) also lies in the interior of the other of C_1, C_2. In this case clearly the polygons C_1 and C_2 must meet at some point between (in the clockwise sense) p_1 and p_2 (with this order). Say p_3 is the closest to p_1 such point. Then clearly $f(p_3)$ is a vertex of one of C_1, C_2 between (in the clockwise sense) p_1 and p_2 (with this order). These parts of C_1, C_2 though, lie outside C_3; the interior of C_3 lie on the other side of the line p_1p_2. Thus $f(p_3)$ is not in $f(C_1 \cap C_2 \cap C_3)$ and the claim is established in all cases. For the side a of C_3 containing p_1, p_2 let us call $q(a)$ a vertex as the one in the claim we just proved. It is easy to see that for distinct sides a, b of C_3 that contain two of the p's, the points q_a, q_b are distinct. Indeed, let a contain p_1, p_2 and b contain p_3, p_4 among the p's. If one of q_a, q_b belongs to one of C_1, C_2 and the other does not belong to it, we are okay. If both q_a, q_b belong to say C_1, then in a clockwise tour around C_1 starting at p_1, we meet p_1, p_2, p_3, p_4 in this order. If not, say the order is p_1, p_3, p_2, p_4. Then the segments p_1p_2, p_3, p_4 intersect at an interior point, since C_1 is a convex polygon. But then the sides a, b of C_3 have a common interior point, a contradiction. So the correct order is p_1, p_2, p_3, p_4. But we know that adding $q(a), q(b)$ in this tour the correct order is $p_1, q(a), p_2, p_3, q(b), p_4$. Thus $q(a), q(b)$ are distinct as claimed.

Now if x of the edges of C_3 contain 1 of the p_i's and y of them contain 2 of the p_i's, then $x + y \leq$ number of sides of C_3, i.e. $x + y \leq n$. The number of points in $C_1 \cap C_2 \cap C_3$ is $x + 2y$. Since f is injective, $x + 2y$ is also the number of q's in $f(C_1 \cap C_2 \cap C_3)$. Also, by the argument in the previous paragraph, we see that for every distinct edge of C_3 containing 2 points we can assign a corresponding distinct q_i outside the image of $f(C_1 \cap C_2 \cap C_3)$. Therefore x is less or equal to the number of q's that do not belong in $f(C_1 \cap C_2 \cap C_3)$. So $(x + 2y) + y$ is at most as much as the number of q's. I.e. $x + 3y \leq 2n$. Adding this with $x + y \leq n$ and dividing by 2, and also taking into account that $x + 2y$ is an integer

$$x + 2y \leq \lfloor \frac{3n}{2} \rfloor$$

Let us now show that this is the best upper bound for every $n \geq 3$. One way (among many) to construct an example is as follows: Construct two regular $n-$gons C_1, C_2 with the same center, such that their intersection points form a regular $2n-$gon. Call the vertices $p_1, p_2, \ldots, p_{2n}$ in a cyclic order. Let the circumcircle of this $2n-$gon be $\mathcal{C}$. Then let the $n-$gon bounded by the lines $p_1p_3, p_5p_7, p_9p_{11}, \ldots$ (including $p_{2k+1}p_1$ in case n is an odd $n = 2k+1$) together with the tangent lines to $\mathcal{C}$ at $p_4, p_8, p_{12}, \ldots$ be C_3. It can easily be checked that $|C_1 \cap C_2 \cap C_3| = \lfloor \frac{3n}{2} \rfloor$. ■

Solution 2: Let A and B be two consequtive points of $C_1 \cap C_2 \cap C_3$ observed in the clockwise direction from a point in the interior of all three n-gons. Let's look for each C_i its section in the clockwise direction between A and B exluding these points. If some two of these sections both do not contain any vertices of their corresponding n-gons, then the segment AB belongs to both n-gons, a contradiction.

Thus at least two of these segments have at least one vertex each, and moreoverthey do not contain the segment. Trivially, two distinct such vertices exist. Since there exist $|C_1 \cap C_2 \cap C_3|$ many conseqtutive points points A and B of $C_1 \cap C_2 \cap C_3$, there should exist at least $2|C_1 \cap C_2 \cap C_3|$ distinct vertices of the three n-gons. Thus $2|C_1 \cap C_2 \cap C_3| \leq 3n$ i.e. $|C_1 \cap C_2 \cap C_3| \leq \lfloor \frac{3n}{2} \rfloor$ since ($|C_1 \cap C_2 \cap C_3|$ is an integer as well).

Actually we can achieve this upper bound by the example given in the Solution 1.