24^{th} BALKAN MATHEMATICAL OLYMPIAD
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Problem 1.
et $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$ and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^0$.

Solution. Let us first denote $\angle BAC = \angle BCA = \alpha$, $\angle CBD = \angle CDB = \beta$.
Part I : Assume $AE = DE$.
By $\triangle EBC$ we have $\angle AEB = \angle DEC = \alpha + \beta$, thus in $\triangle ABE$ we have $\angle ABE = 180^0 - (2\alpha + \beta)$, and in $\triangle CED$ we have $\angle DCE = 180^0 - (\alpha + 2\beta)$. Then by the law of sines in these two triangles we get

$$\frac{AE}{sin(2\alpha+\beta)} = \frac{AB}{sin(\alpha+\beta)} = \frac{CD}{sin(\alpha+\beta)} = \frac{DE}{sin(\alpha+2\beta)}.$$

So $sin(2\alpha + \beta) = sin(\alpha + 2\beta)$ with $0^0 < 2\alpha + \beta, \alpha + 2\beta < 180^0$. So either $2\alpha + \beta = \alpha + 2\beta$ or $2\alpha + \beta + \alpha + 2\beta = 180^0$.

The relation $2\alpha+\beta = \alpha+2\beta$ gives $\alpha = \beta$, which in turn implies $\angle BAD = \angle CDA$, and then $\triangle BAD = \triangle CDA$, from which $AC = BD$, a contradiction.

The relation $2\alpha+\beta+\alpha+2\beta = 180^0$ implies $\alpha+\beta = 60^0$. Then $\angle BAD+\angle ADC = \alpha + \angle EAD + \beta + \angle EDA = \alpha + \beta + \angle AEB = 2(\alpha + \beta) = 120^0$.

Part II : Assume $\angle BAD + \angle ADC = 120^0$.

Let S be the intersection point of the lines AB and DC.

As in part I, we have $\angle AEB = \alpha+\beta$. But also $\angle AEB = \angle EAD+\angle EDA$. Thus $2\angle AEB = \alpha + \beta + \angle EAD + \angle EDA = \angle BAD + \angle ADC = 120^0$. i.e $\angle AEB = 60^0$. But $\angle S$ is also 60^0. So $SBEC$ is cyclic. Thus $\angle BSE = \angle BCA = \alpha = \angle SAC$. So $EA = ES$. Similarly $ED = ES$, and the desired result follows. ■

Remark. We can avoid the use of trigonometry in part I, as follows: The triangles BAE and CDE have two pairs of equal sides and their angles AEB, CEC opposite to the sides of one of these pairs also equal. By a well known theorem on the congruence of two triangles, we know then that the angles ABE and DCA opposite to the sides of the other pair are either equal or they add up to 180^0, etc.