24th BALKAN MATHEMATICAL OLYMPIAD
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Problem 3. Find all positive integers n such that there is a permutation σ of the set $\{1,2,...,n\}$, for which $\sqrt{\sigma(1)+\sqrt{\sigma(2)+\sqrt{...+\sqrt{\sigma(n)}}}}$ is a rational number.

Note: A permutation of the set $\{1,2,...,n\}$ is a one-to-one function of this set to itself.

Solution. For some $n \in N$, let $\sqrt{\sigma(1)+\sqrt{\sigma(2)+\sqrt{...+\sqrt{\sigma(n)}}}} = r_1 \in \mathbb{Q}$. Squaring both sides of the equation we get that $\sqrt{\sigma(2)+\sqrt{\sigma(3)+\sqrt{...+\sqrt{\sigma(n)}}}}$ is also rational. Using the same reasoning recursively, we get that for every $k \in \{1,\ldots,n\}$, $\sqrt{\sigma(k)+\sqrt{\sigma(k+1)+\sqrt{...+\sqrt{\sigma(n)}}}}$ is rational as well. Knowing that the square root of a positive integer is either integer or irrational, we have that $\sqrt{\sigma(n)}$ is integer. Similarly, we get that $\sqrt{\sigma(k)+\sqrt{\sigma(k+1)+\sqrt{...+\sqrt{\sigma(n)}}}}$, for every $k \in \{1,\ldots,n\}$, is integer. Note that for $k=1$ we get $r_1 \in \mathbb{N}$.

We define a_k as $a_k = \underbrace{\sqrt{n+\sqrt{n+\sqrt{...+\sqrt{n}}}}}_{k}$, for all $k \geq 1$. It is easy to prove by induction that $a_k < \sqrt{n}+1$, for every $k \geq 1$. Therefore, we have $\sqrt{\sigma(1)+\sqrt{\sigma(2)+\sqrt{...+\sqrt{\sigma(n)}}}} < a_n < \sqrt{n}+1$, implying $r_1 < \sqrt{n}+1$.

Let ℓ be the positive integer that satisfies $\ell^2 \leq n < (\ell+1)^2$. For some i, $1 \leq i \leq n$, we have $\sigma(i) = \ell^2$. We distinguish two cases:

<u>*First case:*</u> $i \neq n$.

Then we have $\ell < \sqrt{\ell^2+\sqrt{\sigma(i+1)+\sqrt{...+\sqrt{\sigma(n)}}}} < \sqrt{n}+1 < \ell+2$, implying

$\sqrt{\ell^2 + \sqrt{\sigma(i+1) + \sqrt{... + \sqrt{\sigma(n)}}}} = \ell + 1$. But then it follows that

$$2\ell + 1 = \sqrt{\sigma(i+1) + \sqrt{... + \sqrt{\sigma(n)}}} < \sqrt{n} + 1 < \ell + 2,$$

giving $\ell < 1$. A contradiction.

Second case: $i = n$.

For $\ell > 1$, $\ell^2 - 1$ belongs to the set $\{\sigma(1), ..., \sigma(n-1)\}$. Let $j < n$ be such that $\sigma(j) = \ell^2 - 1$. Similarly to the first case, we have

$$\ell < \sqrt{\ell^2 - 1 + \sqrt{\sigma(j+1) + \sqrt{... + \sqrt{\ell^2}}}} < \sqrt{n} + 1 < \ell + 2,$$

implying $\sqrt{\ell^2 - 1 + \sqrt{\sigma(j+1) + \sqrt{... + \sqrt{\ell^2}}}} = \ell + 1$, and

$$2\ell + 2 = \sqrt{\sigma(j+1) + \sqrt{... + \sqrt{\ell^2}}} < \sqrt{n} + 1 < \ell + 2,$$

a contradiction.

If $\ell = 1$, then $n \in \{1, 2, 3\}$. Checking through all the possibilities, it is easy to see that for $n = 1$ and $n = 3$ there exist permutations that satisfy the initial condition. Namely, for $n = 1$ we have $\sqrt{1} = 1$, and for $n = 3$, we have $\sqrt{2 + \sqrt{3 + \sqrt{1}}} = 2$. For $n = 2$ there is no such permutation.