Tuymaada Yakut Olympiad 2007

Day 1

$\boxed{1}$ Positive integers $a < b$ are given. Prove that among every b consecutive positive integers there are two numbers whose product is divisible by ab.

$\boxed{2}$ Two polynomials $f(x) = a_{100}x^{100} + a_{99}x^{99} + \cdots + a_1x + a_0$ and $g(x) = b_{100}x^{100} + b_{99}x^{99} + \cdots + b_1x + b_0$ of degree 100 differ from each other by a permutation of coefficients. It is known that $a_i \neq b_i$ for $i = 0, 1, 2, \ldots, 100$. Is it possible that $f(x) \geq g(x)$ for all real x?

$\boxed{3}$ AA_1, BB_1, CC_1 are altitudes of an acute triangle ABC. A circle passing through A_1 and B_1 touches the arc AB of its circumcircle at C_2. The points A_2, B_2 are defined similarly. Prove that the lines AA_2, BB_2, CC_2 are concurrent.

$\boxed{4}$ Determine maximum real k such that there exist a set X and its subsets Y_1, Y_2, ..., Y_{31} satisfying the following conditions: (1) for every two elements of X there is an index i such that Y_i contains neither of these elements;

(2) if any non-negative numbers α_i are assigned to the subsets Y_i and $\alpha_1 + \cdots + \alpha_{31} = 1$ then there is an element $x \in X$ such that the sum of α_i corresponding to all the subsets Y_i that contain x is at least k.

Day 2

1. What minimum number of colours is sufficient to colour all positive real numbers so that every two numbers whose ratio is 4 or 8 have different colours?

2. Point D is chosen on the side AB of triangle ABC. Point L inside the triangle ABC is such that $BD = LD$ and $\angle LAB = \angle LCA = \angle DCB$. It is known that $\angle ALD + \angle ABC = 180^\circ$. Prove that $\angle BLC = 90^\circ$.

3. Several knights are arranged on an infinite chessboard. No square is attacked by more than one knight (in particular, a square occupied by a knight can be attacked by one knight but not by two). Sasha outlined a 14×16 rectangle. What maximum number of knights can this rectangle contain?

4. Prove that there exists a positive c such that for every positive integer N among any N positive integers not exceeding $2N$ there are two numbers whose greatest common divisor is greater than cN.